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Chebyshev Interpolation with Approximate Nodes of Unrestricted Multiplicity

GILBERT STENGLE

Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015, U.S.A.

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1. INTRODUCTION

We investigate high order polynomial interpolation of smooth functions on a finite interval. Our main result is an asymptotic estimate relating differentiability properties of an interpolated function to a condition on the asymptotic distribution of nodes, without restriction on multiplicity, which will eliminate the Runge phenomenon. Roughly speaking, the smoother the function, the less critical the location of nodes. Our main result, Theorem 5.1 below, is a quantified relation of this kind. Perhaps the simplest example of this phenomenon is the following. Let T_n denote the nth Chebyshev polynomial.

EXAMPLE 1.1. If f is real analytic on $[-1, 1]$, and f_n is the Hermite interpolant to f at nodes obtained by rounding the roots of T_n to a sufficiently fine fixed precision (depending on f but not on n), then the sequence f_n converges geometrically to f.

This is not hard to verify directly. It will also follow as a special case of our results. We note that the scheme here amounts, for large n, to interpolating f at nodes of high multiplicity lying in a sufficiently fine regularly spaced grid. Since our main result is somewhat technical, we illustrate its meaning with a more easily stated application to interpolation of smooth functions with such a scheme. For this we require two definitions; first, a measure of the smoothness of f (which is the fundamental datum for our entire investigation), and second, a precise description of such an interpolation scheme.

DEFINITION 1.2. Let φ be a convex function on $[0, \infty]$ satisfying $\varphi(0)=0$, $\varphi'(U) \geq \log U + 2$. Let B_φ be the class of smooth functions of

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period 2 satisfying derivative estimates $\|f^{(m)}\|_\infty \leqslant \exp \varphi(m)$. Let φ^* be the convex conjugate function of φ defined by $\varphi^*(V) = \sup_U \{UV - \varphi(U)\}$.

DEFINITION 1.3. Given positive integers N and n, let $x_k = -1 + (2k+1)/N$, $k = 0, 1, ..., N-1$. Let m_k be the number of zeros of T_n, the nth Chebyshev polynomial, in the interval $[x_k - 1/N, x_k + 1/N]$. Given a smooth function f on $[-1, 1]$ let $P_{n,N}(f)$ be the polynomial of least degree interpolating f at each x_k to order at least m_k.

THEOREM 1.4. *Let $f \in B_\varphi$. Then there are positive constants c and C such that as $n \to \infty$*

$$|f - P_{n,N}(f)| \leqq \exp\{-c\varphi^*(\log n)\}$$

provided N varies with n in such a way that

$$N \geqq C\{n(\log n - \log(\varphi^{*\prime}\{\log n\})/\varphi^*\{\log n\})\}^2.$$

The sense of this is nicely illustrated by using it to check Example 1.1. By Cauchy's estimate, f is real analytic if and only if it satisfies derivative estimates of the form $|f^{(m)}| \leqq LC^m m!$. Stirling's formula shows that this corresponds to $\varphi(m)$ of the form $m \log m + am$. This gives $\varphi^*(V) = \exp\{V - 1 - a\}$ and simple calculation shows that the constraint on N reduces to $N \geqq C\{(1+a)\exp(1+a)\}^2$. Since $\varphi^*(\log n) = n \exp\{-1-a\}$ is linear in n the asserted error estimates also ensure geometric convergence.

In Section 2 we discuss the meaning of φ and B_φ. Sections 3 and 4 [1, Chap. 12] we discuss the meaning of analytic data. In Section 5 we prove our main results. Section 6 further discusses our hypotheses and results and raises some open questions.

2. A QUANTIFICATION OF SMOOTHNESS

For technical convenience we suppose that f is a smooth function of period 2 with mean 0. A function on a finite interval always has a periodic extension although there are classes of analytic or quasi-analytic data which do not contain partitions of unity so that this extension might not be possible without increasing derivative estimates for the extended function. Altogether, the use of periodic data evades these fine points and permits a single result which is sharp for a wide range of data. Suppose then that $f(x) = \sum_{k \neq 0} a_k \exp i\pi kx$.

DEFINITION 2.1. For $\alpha \geqslant 0, f \not\equiv 0$ let

$$\varphi(\alpha) = \log\left(\sum |a_k|\, |\pi k|^\alpha\right).$$

Then φ is convex for $\alpha \geqq 0$. For

$$\varphi\left(\frac{\alpha+\beta}{2}\right) = \log\left\{\sum |a_k|^{1/2}\, |\pi k|^{\alpha/2}\, |a_k|^{1/2}\, |\pi k|^{\beta/2}\right\}$$

$$\leqslant \log\left\{\left(\sum |a_k|\, |\pi n|^{\alpha}\right)^{1/2} \left(\sum |a_k|\, |\pi k|^{\beta}\right)^{1/2}\right\}$$

$$\leqslant \frac{1}{2}\,\varphi(\alpha) + \frac{1}{2}\,\varphi(\beta).$$

Also $\|f^{(m)}\|_\infty \leqslant \exp \varphi(m)$. (In fact $\|f^{(\alpha)}\|_\infty \leqslant \exp \varphi(\alpha)$ for real $\alpha \geqslant 0$ if we understand $f^{(\alpha)}$ to be the Riemann–Liouville derivative of order α.) Again for technical convenience, define $\varphi(\alpha) = +\infty$ for $\alpha < 0$. Then φ is an extended-real-valued function on the line for which the relation $\varphi^*(V) = \sup_U (UV - \varphi(U))$ defines φ^* as another such function. Finally, since all our analysis is linear and homogeneous, replacing f by a suitable constant multiple of f ensures that $\varphi(0) = 0$.

LEMMA 2.2. *$V = o(\varphi^*(V))$ as $V \to \infty$.*

Proof. By its definition φ is a lower-semi-continuous convex function. Such functions satisfy the duality relation $\varphi^{**} = \varphi$, that is, $\sup_V (UV - \varphi^*(V)) = \varphi(U)$. Hence for large V, $V \leqslant (1/U)\, \varphi^*(V)$ or $\limsup_{V\to\infty} V/\varphi^*(V) \leqslant 1/U$ which implies $\limsup_{V\to\infty} V/\varphi^*(V) = 0$.

Our definition of B_φ is formulated to exclude two kinds of data which fit easily into this framework but which are unnatural from the standpoint of our main question. On one hand there is C^k data which can be described by defining $\varphi(\alpha) = +\infty$ for $\alpha > k$. However, such data does not allow the increasing multiple use of nodes which we do not wish to preclude. On the other hand the requirement $\varphi'(U) \geqslant \log U + 2$ somewhat arbitrarily marks a dividing line between functions merely holomorphic in a certain strip about the real axis and functions with smaller φ's enjoying even stronger regularity properties (entire functions, etc.). Our hypotheses exclude the possibility of using sharp derivative estimates for such functions. However, since our results do not depend on these finer properties we save just enough information in these cases to obtain simple unified conclusions.

3. REMAINDER ESTIMATES

We parameterize the general interpolation scheme by associated distribution functions and logarithmic potentials.

DEFINITION 3.1. Let F be a distribution function assigning measure 1 to the interval $[0, \pi]$. Define $U_F(x)=\int_0^\pi \log 2|x-\cos\theta|\, dF$. In the case that ndF is discrete assigning integer weights, define $T_F(x)$ to be the polynomial of the form $2^n x^n + \cdots$ of which $\exp\{nU_F(x)\}$ is the modulus.

For example, the following distributions describe the scheme of Definition 1.3. Let $[\cdot]$ denote the greatest integer function.

LEMMA 3.2. *For* $0 \leqslant \theta \leqslant \pi$ *let*

$$F_{n,N}(\theta)=\frac{1}{n}\left[\frac{n}{\pi}\cos^{-1}\frac{2}{N}\left\{\left[\frac{N(1+\cos\theta)}{2}+\frac{1}{2}\right]-1\right\}+\frac{1}{2}\right].$$

Then each x_k *is a root of* $T_{F_{N,n}}$ *with multiplicity* m_k.

Proof. Let $F=F_{N,n}$. It is clear that the measure ndF assigns weight n to $[0, \pi]$ and has integer jumps and so defines a polynomial T_F. F can jump up only if $[N(1+\cos\theta)/2+1/2]$ jumps down from $k+1$ to k as θ passes $\cos^{-1}\{-1+(2k-1)/N\}=\cos^{-1} x_k$. Thus the zeros of T_F lie among the x_k. The jump in nF at $\cos^{-1} x_k$ (possibly 0) is

$$\begin{aligned} &nF(\cos^{-1} x_k)-nF(\{\cos^{-1} x_k\}^-)\\ &\quad=\left[\frac{1}{2}+\frac{n}{\pi}\cos^{-1}\left(-1+\frac{2(k-1)}{N}\right)\right]\\ &\qquad-\left[\frac{1}{2}+\frac{n}{\pi}\cos^{-1}\left(-1+\frac{2k}{N}\right)\right]\\ &\quad=\left[\frac{1}{2}+\frac{n}{\pi}\cos^{-1}\left(x_k-\frac{1}{N}\right)\right]-\left[\frac{1}{2}+\frac{n}{\pi}\cos^{-1}\left(x_k+\frac{1}{N}\right)\right]. \end{aligned}$$

But this jump counts the number of j's for which $x_k-1/N \leqslant \cos(\pi/n)(j-1/2)\} < x_k+1/N$. This is the number of zeros of the nth Chebyshev polynomial in $[x_k-1/N, x_k+1/N)$ which, by definition, is m_k.

DEFINITION 3.2. Let $P_F(f)$ be the Hermite interpolant of f at the zeros of T_F. Let $R_F(f)=f-P_F(f)$.

We next obtain a preliminary remainder estimate based on the representation of $R_F(f)$ for analytic f as a contour integral

$$R_F(f)=\frac{1}{2\pi i}\,T_F(x)\int_\Gamma (t-x)^{-1}\;T_F^{-1}(t)\,f(t)\,dt,$$

where Γ is a curve encircling $[-1, 1]$.

LEMMA 3.3. *Let $f \in B_\varphi$. Let Γ_m be a sequence of curves encircling* $[-1, 1]$ *of uniformly bounded length. Then*

$$\log|R_F(f)(x)| \leqslant \sup_{m \geqslant 0} \max_{t \in \Gamma_m} \{n(U_F(x) - U_F(t)) + 2\log m - \log|t-x| + \pi m(\operatorname{Im} t) - \varphi^*(\log \pi m) + C\}$$

for some constant C.

Proof. Our hypotheses ensure that

$$R_F(f) = \sum_{k \neq 0} a_m R_F(\exp ik\pi x)$$

since the Lagrange interpolation formula with remainder shows that, whenever n,

$$R_F(f)(x) = \frac{1}{n!\,2^n} T_F(x) f^{(n)}(\xi).$$

Hence R_F is a bounded mapping from C^n to C. But $f \in B_\varphi$ ensures that the Fourier series of f converges rapidly to f in each C^n. Hence

$$R_F(f) = \sum_{m \neq 0} a_m R_F(\exp im\pi x).$$

The relation $\sum |a_m|\,|\pi m|^\alpha = \exp \varphi(\alpha)$ implies

$$|a_m| \leqslant \exp(\varphi(\alpha) - \alpha \log \pi|m|).$$

This gives a one parameter family of upper bounds of which the sharpest is, by the definition of ( )*,

$$|a_m| \leqslant \exp\{-\varphi^*(\log \pi|m|)\}.$$

Hence, representing R_F's as integrals over Γ_m,

$$R_F(f) \leqslant \frac{1}{2\pi} \sum_{m \neq 0} \frac{1}{m^2} |T_F(x)| \int_{\Gamma_m} |t-x|^{-1} \cdot |T_F^{-1}(t)| \exp|\operatorname{Im} t|\,|dt| \exp\{2\log m - \varphi^*(\log \pi m)\}.$$

Using $\log|T_F| = nU_F$ and factoring out the supremum indicated in the conclusion from the convergent sum $\sum 1/m^2$ completes the estimate.

To proceed we require upper estimates for $U_F(x)$, lower estimates for

$U_F(t)$, and a choice for the sequence Γ_m which can also depend on n, x, or anything else that is technically useful.

4. ESTIMATES FOR U_F

We recall some important properties of Chebyshev interpolation (Krylov, [1, Chap. 12]) which account for its behavior and underlie our further estimates. Consider the distribution functions given on $[0, \pi]$ by $F_n = (1/n)[n\theta/\pi + 1/2]$. As $n \to \infty$ these tend to the uniform distribution θ/π. The associated polynomials T_{F_n} are just the Chebyshev polynomials and the corresponding potentials U_{F_n} tend to the limit

$$U_{\theta/\pi} = \frac{1}{\pi} \int_0^\pi \log 2|x - \cos\theta| \, d\theta$$

$$= \log|x + (x^2 - 1)^{1/2}|.$$

This potential vanishes on $[-1, 1]$ and the nearby equipotential curves are narrow ellipses approximating $[-1, 1]$. These ellipses bound regions which are the natural domains of good approximation for Chebyshev interpolants of analytic functions. The potentials U_{F_n} have equipotential curves which approximate these ellipses. (Warner [2] contains vivid pictures showing this approximation).

Our subsequent analysis is an asymptotic quantification of the following qualitative scheme. First, the $F_n(\theta)$ converge rapidly to θ/π so that the equipotential curves of the U_{F_n} rapidly approximate narrow curves surrounding $[-1, 1]$. Second, a function on $[-1, 1]$ enjoying some smoothness properties (say a Dini–Lipschitz condition) is the limit of a sequence of functions, f_m, which are analytic and uniformly bounded on a corresponding sequence of complex neighborhoods $\mathcal{U}_m$ of $[-1, 1]$. Naturally, if f is not analytic, these complex neighborhoods must shrink as the degree of approximation increases (otherwise the limit function f would be analytic too). We wish to choose $m(n)$ large enough so that we can estimate $R_{F_n}(f) \sim R_{F_n}(f_{m(n)})$ and also small enough so that we can estimate $R_{F_n}(f_{m(n)}) \sim 0$. The difficulty in this latter estimate is that it will obtain on $[-1, 1]$ only if $\mathcal{U}_{m(n)}$ contains an equipotential curve of U_{F_n} surrounding $[-1, 1]$. However, if the $\mathcal{U}_{m(n)}$ shrink too rapidly these curves would all be approximate unions of small circles surrounding the nodes and we could only get estimates on the meager part of $[-1, 1]$ interior to these curves.

These considerations apply with even greater force to our problem since we must analyze the coarser approximations to θ/π given by the distributions $F_{n,N}$ of Lemma 3.1. These include, at one extreme, $F_n = \lim_{n \to \infty} F_{n,N}$ and at the other extreme, F_{n,N_0} for fixed N_0, which merely approximates

θ/π to a certain fixed degree without actually converging to it. Our results, beyond the obvious requirement that multiply confluent nodes make sense only with smooth data, can be accounted for by observing that slow (or non-) convergence of $F_{n,N}$ to θ/π forces us to require slow (or non-) shrinkage of the domains $\mathcal{U}_m$ which is only possible for functions enjoying sufficient smoothness properties.

We now obtain upper estimates for U_F on $[-1, 1]$ and lower estimates on ellipses surrounding $[-1, 1]$ given parametrically in the form $\{\cos(\alpha - i\delta) \mid 0 \leqslant \alpha < 2\pi\}$. For fixed δ these are the level curves of the decisive potential $U_{\theta/\pi}$ along which its value is just δ.

DEFINITION 4.1. Let $\varepsilon(F) = \sup_{[0,\pi]} |F(\theta) - \theta/\pi|$.

LEMMA 4.2. *For* $0 < \delta \leqslant 1$, *there is a positive constant* A *such that*

$$|U_F(\cos(\alpha - i\delta)) - \delta| \leqslant A\varepsilon(F)\{\log\{1/\delta\} + 1\}.$$

Proof. Since $U_{\theta/\pi}(\cos\{\alpha - i\delta\}) = \delta$,

$$\begin{aligned}\pi&|U_F(\cos(\alpha - i\delta)) - \delta| \\ &= \left| \int_0^\pi \log\{2|\cos(\alpha - i\delta) - \cos\theta|\}\, d\{F(\theta) - \theta/\pi\} \right| \\ &= \left| \int_0^\pi \log|\cos(\alpha - i\delta) - \cos\theta|\, d\{F(\theta) - \theta/\pi\} \right|.\end{aligned}$$

Since $F(\theta) - \theta/\pi$ vanishes at the endpoints, integrating by parts and estimating we find

$$\pi|U_F(\cos(\alpha - i\delta)) - \delta| \leqslant \varepsilon(F)\, \mathrm{var}_\theta\{|\log|\cos(\alpha - i\delta) - \cos\theta||\}.$$

To estimate the indicated variation over θ we observe that $|\cos(\alpha - i\delta) - \cos\theta|^2 = (\cos\alpha\cosh\delta - \cos\theta)^2 + \sin^2\alpha\sinh^2\delta$ is either monotonic if $\cos^2\alpha\cos^2\delta \geqslant 1$ or else has a minimum for $\cos^\theta = \cos\alpha\cosh\delta$. Thus in either case $\mathrm{var}_\theta \leqslant 2(\max_\theta - \min_\theta)$. But some algebra shows that $|\cos(\alpha - i\delta) - \cos\theta|^2 = (\cos\alpha - \cos\theta)^2 + 4(\cosh^2\delta - \cos\theta\cos\alpha)\sinh^2(\delta/2)$. For this an upper estimate is $4(1 + [\cosh 1 + 1]^2)\sinh^2\frac{1}{2}$ and a lower estimate is $4(\cosh^2(\delta/2) - 1)\sinh^2(\delta/2) = 4\sinh^4(\delta/2)$. Combining these estimates with the bound $\sinh(\delta/2) \geqq c\delta$ gives the conclusion.

LEMMA 4.3. *For* $x \in [-1, 1]$ *there is a positive* B *such that*

$$U_F(x) \leqslant B\varepsilon(F).$$

Proof. The potential $U_F - U_{\theta/\pi}$ is subharmonic and agrees with U_F on $[-1, 1]$. Hence, by the maximum principle, an upper bound for $U_F - U_{\theta/\pi}$ on any curve surrounding $[-1, 1]$ will be an upper bound for U_F on $[-1, 1]$. Choosing $\delta = 1$ in Lemma 4.1 gives such an estimate.

We now combine our lemmas into a remainder estimate.

LEMMA 4.4. *Let $f \in B_\varphi$. Let $\varepsilon = \varepsilon(F) = \sup_\theta |F(\theta) - \theta/\pi|$. Let $\delta(m)$ be a sequence of numbers in $(0, 1]$. Then there is a positive constant C such that*

$$\log|R_F(f)| \leqslant \inf_{\delta(\cdot)} \sup_{m>0}\{-n\delta(m) + 2m\delta(m) + 2\log m + (Cn\varepsilon + 2)\log\{1/\delta(m)\} + Cn\varepsilon - \varphi^*(\log \pi m) + C\}.$$

Proof. We combine the estimate of Lemma 3.3 with those of 4.2 and 4.3. Let Γ_m in 3.3 be the ellipse $t = \cos(\alpha - i\delta(m))$. Then, by an estimate from the proof of Lemma 4.2, on Γ_m

$$|t - x|^{-1} \leqslant \left\{2\sinh^2 \frac{\delta(m)}{2}\right\}^{-1} \leqslant 2\delta(m)^{-2}.$$

Also $|\mathrm{Im}\, t| = |\sin\alpha \sinh \delta(m)| \leqslant 2\delta(m)$ and, by the conclusion of Lemma 2.2, $-U_F(\alpha - i\delta) \leqq -\delta + B\varepsilon(A \log 1/\delta(m+1))$. Combining these inequalities gives the stated estimate.

We have reduced the estimation of $R_F(f)$ to a kind of optimization problem, namely that of choosing $\delta(\cdot)$ to estimate the infimum indicated in this lemma.

The following estimate is a key to bringing out a certain explicit dependence on φ^* in our main results.

LEMMA 4.5. *Let φ be smooth, convex, satisfying $\varphi'(0) \geqslant \log U + 2$, $\varphi(0) = 0$. Then*

(a) $\varphi^{*\prime}(V) \leqslant \exp(V - 2)$,

(b) *for large V*

$$\varphi^*(V)/(V - \log \varphi^{*\prime}(V)) < \varphi^{*\prime}(V).$$

Proof. The mutually conjugate functions φ and φ^* are related by

$$\varphi^*(V) = U^*(V)V - \varphi(U^*(V)), \qquad U^*(V) = \varphi^*(V)$$
$$\varphi(V) = UV^*(U) - \varphi^*(V^*(U)), \qquad V^*(U) = \varphi'(U).$$

From $V^*(U)=\varphi'(V)$ we have $V^*(U)\geqslant \log U+2$ which implies $U^*(V)=\varphi^*(V)\leqslant \exp(V-2)$. Elementary calculus shows

$$\varphi^*(V)=V\varphi^{*\prime}(V)-\int_0^V s\varphi^{*\prime\prime}(s)\,ds+\varphi^*(0).$$

By estimate (a) $s\geqslant 2+\log \varphi^{*\prime}(s)$. Also $\varphi^{*\prime}\geqslant 0$. Hence

$$\begin{aligned}\varphi^*(V)&\leqslant V\varphi^{*\prime}(V)-\int_0^V [2+\log \varphi'(s)]\,\varphi^{*\prime\prime}(s)\,ds+\varphi(0)\\ &\leqslant V\varphi^{*\prime}(V)-\varphi^{*\prime}(V)\log {}^{*\prime\prime}(V)-\varphi^{*\prime}(V)+C.\end{aligned}$$

But $\varphi^{*\prime}$ is large for large V. Hence for large V

$$\varphi^*(V)<\{V-\log(\varphi^{*\prime}(V))\}\,\varphi^{*\prime}(V).$$

5. Precision Requirements for Nodes

The following is our main result. It essentially estimates how much precision in the array of Chebyshev nodes is needed to eliminate the Runge phenomenon.

THEOREM 5.1. *Let G_n be a sequence of distribution functions assigning measure* 1 *to* $[0,\pi]$. *Let f belong to B_φ. Let $\varepsilon(G_n)=\sup_\theta|G_n(\theta)-\theta/\pi|$. Then there exist positive constants a and A such that*

$$R_{G_n}(f)\leqq \exp\{-a\varphi^*(\log n)\}$$

provided that

$$\varepsilon(G_n)\cdot n\{\log n-\log \varphi^{*\prime}(\log n)\}/\varphi^*(\log n)>A.$$

Proof. In the estimate of 4.4 let $\varepsilon'=C\varepsilon(G_n)$ and

$$\delta(m)=\min\left\{\frac{\varepsilon'+2}{2m-n},\,1\right\}.$$

We can drop the requirement that m be an integer at the expense, perhaps, of weakening the upper estimate. Then for $2m\leqslant n+n\varepsilon'+2$, $\delta(m)=1$. Let $E_1=2m-n+2\log m+n\varepsilon'-\varphi^*(\log \pi m)$. Since

$$\frac{\partial E_1}{\partial m}=2+\frac{2}{m}-\frac{\varphi^{*\prime}(\log \pi m)}{m},$$

by Lemma 4.5a

$$\frac{\partial E_1}{\partial m} \geqslant 2 + \frac{2}{m} - \frac{\pi m e^{-2}}{m} > 0.$$

Hence E_1 is monotonic increasing and its maximum occurs at the upper endpoint of its domain where

$$E_1 \leqslant 2 + n\varepsilon' + 2\log\left(\frac{n}{2} + \frac{n\varepsilon'}{2} + 1\right)$$
$$+ n\varepsilon' - \varphi^*\left(\log \pi \left(\frac{n}{2} + \frac{n\varepsilon'}{2} + 1\right)\right) + C$$
$$\leqslant 2n\varepsilon' + 2\log n - \varphi^*(\log n) + C.$$

Notice that we are washing $\pi/2$ out of our estimates as a second order feature which, in any case, depends on our conventional choice of an interval of length 2. Since $\varphi^*(U)$ grows more rapidly than any linear function, for n sufficiently large, $E_1 \leqslant 2n\varepsilon' - \frac{1}{2}\varphi^*(\log n)$. If $n\varepsilon'/\varphi^*(\log n)$ is small, this implies that $E_1 = -O\{\varphi^*(\log n)\}$. By Lemma 4.5 $\varphi^*(U) \leqslant \exp(U-2)$. Hence

$$U - \log \varphi^{*\prime}(\log U) \geqslant (1 - e^{-2})\, U.$$

Thus the stricter condition

$$\varepsilon' n(\log n - \log \varphi^{*\prime}(\log n))/\varphi^*(\log n) < \tfrac{1}{25}$$

also ensures that $E_1 = -O\{\log \varphi^*(\log n)\}$.

Similarly, for $2m > n + n\varepsilon' + 2$ let

$$E_2 = (n\varepsilon' + 2)\left(1 + \log\frac{2m-n}{n\varepsilon'+2}\right) + 2\log m + n\varepsilon' - \varphi^*(\log \pi m).$$

Then for large n

$$E_2 \leqslant n\varepsilon' \log\frac{2m-n}{n\varepsilon'+2} + 4\log m + 3n\varepsilon' + 2 - \varphi^*(\log \pi m)$$
$$\leqslant n\varepsilon'\left(3 + \log\frac{2m-n}{n\varepsilon'+2}\right) - \frac{1}{2}\varphi^*(\log \pi m).$$

Call this bound E_3. Then E_3 has an extremum either at $m = 1/2 + n\varepsilon'/2 + 1$ or at a critical point. Critical points satisfy

$$\frac{2n\varepsilon'}{2m-n} - \frac{1}{2}\frac{\varphi^{*\prime}(\log \pi m)}{m} = 0$$

or

$$m = \frac{n/2}{1 - \dfrac{n\varepsilon'}{\varphi^*(\log \pi m)}} < \frac{n/2}{1 - \dfrac{n\varepsilon'}{\varphi^*(\log(\pi/2)n)}} < \frac{n/2}{1 - \dfrac{n\varepsilon'}{\varphi^*(\log n)}}.$$

If $n\varepsilon'/\varphi^{*\prime}(\log n) < \frac{1}{2}$, then any critical point of E_3 must satisfy $m < 2n$. By Lemma 4.5, this condition is ensured by the given hypotheses. Hence the maximum of E_3 occurs in the domain $n/2 \leqslant m < n$. At the lower endpoint E_3 reduces to E_1. At any critical point

$$\begin{aligned} E_3 &\leqslant n\varepsilon'\left(3 + \log \frac{2m-n}{n\varepsilon'}\right) - \frac{1}{2}\varphi^*\left(\log \frac{\pi}{2} n\right) \\ &\leqslant n\varepsilon'(3 + \log(4m/\varphi^{*\prime}(\log \pi m)) - \frac{1}{2}\varphi^*\left(\log \frac{\pi}{2} n\right) \\ &\leqslant n\varepsilon'(3 + \log(4n/\varphi^{*\prime}(\log \pi n/2))) - \frac{1}{2}\varphi^*\left(\log \frac{\pi}{2} n\right) \\ &\leqslant (3 + \log 4)\, n\varepsilon' + n\varepsilon'(\log n - \varphi^{*\prime}(\log n)) - \frac{1}{2}\varphi^*(\log n). \end{aligned}$$

Again the hypothesized estimate on $\varepsilon = \varepsilon(G_n)$ ensures that

$$E_3 = -O(-\varphi^*(\log n)).$$

Combining the above estimates for E_1 and E_3 with $\log|R_{F_n}(f)| \leqslant \max(\sup_m E_1, \sup_m E_3)$, completes the proof.

Theorem 1.4 stated in Section 1 is a direct consequence.

Proof. The interpolants $P_{n,N}(f)$ correspond to the distributions $F_{n,N}$ given in Lemma 3.1. These satisfy

$$F_{n,N}(\theta) = \theta/\pi + O\left(\frac{1}{n}\right) + O\left(\frac{1}{\sqrt{N}}\right).$$

Hence $\varepsilon(F_{n,N}) = O(1/n) + O(1/\sqrt{N})$. By Theorem 1 it suffices that

$$\max\left(\frac{1}{n}, \frac{1}{\sqrt{N}}\right) n\{\log n - \log \varphi^{*\prime}(\log n)\}/\varphi^*(\log n)$$

be sufficiently small. This is equivalent to the asserted growth condition on N.

6. Remarks and Open Questions

We can illustrate the meaning of φ as a datum by posing the following question: how many derivatives have we actually used in interpolating with the $P_{n,N}(f)$ of Theorem 1.4.?

There are three ways to understand this question.

(1) Of course we have assumed infinitely differentiable data and even to obtain an infinite sequence of approximations we might need all derivatives of f.

However, it is also significant to ask how many derivatives do we need:

(2) to compute $P_{n,N}(f)$?

(3) to estimate $R_{n,N}(f)$?

We sketch answers to these questions.

First, Theorem 1 requires that

$$N = O(n\{\log n - \log \varphi^{*\prime}(\log n)\}/\varphi^*(\log n))^2.$$

It is plain that the heaviest multiple use of a node occurs near the endpoints. Explicitly

$$\begin{aligned} m_1 &= \left[\frac{1}{2} + \frac{n}{\pi}\cos^{-1}(-1)\right] - \left[\frac{1}{2} + \frac{n}{\pi}\cos^{-1}\left(-1 + \frac{2}{N}\right)\right] \\ &\sim \frac{n}{\pi}\left\{1 - \cos^{-1}\left(1 - \frac{2}{N}\right)\right\} \\ &\sim \frac{2}{\pi}\frac{n}{\sqrt{N}} = O\{\varphi^*(\log n)/(\log n - \log \varphi^*(\log n))\}. \end{aligned}$$

Thus simply to define or compute $P_{n,N}(f)$ we require $O\{\varphi^*(\log n)/(\log n - \log \varphi^{*\prime}(\log n))\}$ derivatives.

On the other hand, the number of derivatives we have used to estimate $R_{n,N}(f)$ appears implicitly in the rate of convergence $\exp\{-O(\varphi^*(\log n))\}$. The convergence exponent $\varphi^*(\log n)$ can be expressed (as in the proof of Lemma 4.5) in terms of φ by

$$\varphi^*(\log n) = (\log n)\, U^*(\log n) - \varphi(U^*(\log n)),$$

where $U^*(\log n) = \varphi^{*\prime}(\log n)$. The argument of φ is roughly the number of derivatives essential to the estimate. Thus $O(\varphi^{*\prime}(\log n))$ derivatives are required to estimate $R_{n,N}(f)$. By Lemma 4.5b for large U, $\varphi^*(U)/(U - \log \varphi^{*\prime}(U)) < \varphi^{*\prime}(U)$. Thus these estimates check that more smoothness is needed to estimate the remainder than to compute the interpolating polynomial (obviously the case for ordinary Chebyshev interpolation where any function can be used to define interpolants but some smoothness is needed to estimate remainders).

It seems unlikely that the interpolants $P_{n,N}$ as we have defined them are the best possible with the constraint that interpolation nodes are regularly spaced. These would surely result from using the array of nodes corresponding to the polynomials

$$U_{n,N} = 2^{-n} \prod (x - x_k)^{n_k}$$

where

$$n_k \geqslant 0, \qquad \sum n_k = n,$$

and $U_{n,N}$ has the minimum supremum norm among all polynomials of this form. Our polynomials $T_{n,N}$ are a simple guess at an approximation to the $U_{n,N}$. The determination of each $U_{n,N}$ is evidently an integer programming problem. We conclude by asking: is it possible to obtain systematic asymptotic information about the $U_{n,N}$ and use it to improve our estimates by improving the underlying interpolation scheme?

References

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